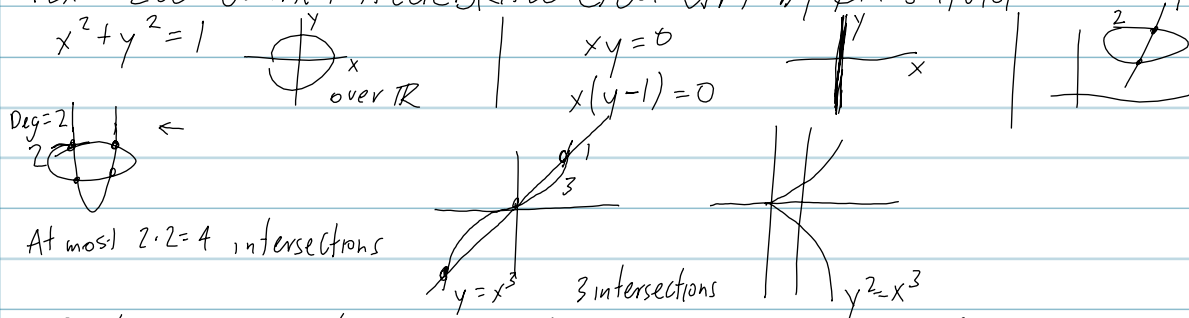


ALGEBRAIC GEOMETRY - LECTURE ONE

9/26/17

TEXT: ELEMENTARY ALGEBRAIC GEOMETRY by Klaus Hulek



Conjecture: The intersection of polynomial curves of degrees d_1 and d_2 has $\leq d_1 d_2$ points. Why don't we get exactly $d_1 d_2$?

Obstacles: (1) look at system $xy=0$ and $x(y-x)=0$. We see the entire y -axis lies on solution set. So \nexists only many intersection points for 2 curves

(2) look at $x^2 + y^2 - 1 = 0$ and $x=2$ over \mathbb{R} there is no intersection. Over \mathbb{C} , \exists 2 solutions: $(2, i\sqrt{3}), (2, -i\sqrt{3})$ (The answer depends on the field.)

(3) look at $x^2 + y^2 = 1$ and $x=1$ over \mathbb{R} Consider the multiplicity of solutions

(4) Consider $x+y=1$ and $x+y=2$ Need to expand and work in projective space....

Coming Bezout's Theorem...

$x^2 + y^2 = 1$ parameterized by $x = \frac{1-t^2}{1+t^2}$ $y = \frac{2t}{1+t^2}$

Handout w/ Fermat's cubic $x^3 + y^3 + z^3 = 1$ and parametrization w/ graph

Algebra \rightarrow Geometry and Geometry to Algebra

y^2 y times anything - Ideal generated by y .

example: y^2 , what points does it vanish on? V is x -axis?

Handout: Review of Algebra: Groups, Rings, Integral domains, ideals, principal ideal.

Ideal generated by: $\mathbb{Z} \langle 4, 6 \rangle = \{4a + 6b \mid a, b \in \mathbb{Z}\} = \langle 2 \rangle$

$\mathbb{Z}[x]$ = ring of all polynomials with integer coefficients. = $\{a_0 + a_1x + \dots + a_nx^n \mid a_i \in \mathbb{Z}, n \geq 0\}$

Ideal: $\langle x \rangle$
 $\langle x^2 \rangle$

$\langle x, 2 \rangle = \{f(x) \in \mathbb{Z}[x] \mid \text{constant term of } f \text{ is even}\}$


\rightarrow not a principal ideal

rings:
 $R[x] \leftarrow$
 $C[x]$

If K , a field, $K[x]$ is a principal ideal domain.
 $R(x)$ is a field $\frac{x-3}{x^2+5}$ $C[x,y,z]$ vs. $C(x,y,z)$
 Definition: A ring R is Noetherian if it satisfies the ascending chain condition (ACC) on ideals; that is, if $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ is an ascending chain of ideals in R , then \exists integer n such that $I_n = I_{n+1} = I_{n+2} = \dots$

$R[x]$ varieties:
 schemes
 $x=0$
 $x^2=0$
 distinguishes these...

Example: $\mathbb{Z}[x]$ $\langle 12 \rangle \subseteq \langle 4 \rangle \subseteq \langle 2 \rangle \subseteq \langle 1 \rangle = \mathbb{Z}$
 also $\langle 12 \rangle \subseteq \langle 6 \rangle \subseteq \langle 3 \rangle \subseteq \langle 1 \rangle = \mathbb{Z}$

sheets
 $y^2 = x^3$

 cusp

THEOREM: The following are equivalent (TFAE) for a ring R :

- (a) R is Noetherian
- (b) R satisfies the maximal condition on ideals; that is, any non-empty set of ideals in R has a maximal element, one not contained in any other ideal of the set.
- (c) Every ideal of R is finitely generated.

Example: $R[x_1, x_2, x_3, x_4, \dots]$ polynomials in infinitely many variables with real coefficients.

$x_1^2 x_4 + x_2^5 + x_3$
 \mathbb{Z}
 $S = \{\langle 2 \rangle, \langle 3 \rangle, \langle 7 \rangle\}$
 R
 I
 J

$\Rightarrow \langle x_1 \rangle \subseteq \langle x_1, x_2 \rangle \subseteq \langle x_1, x_2, x_3 \rangle \subseteq \dots$ not a Noetherian ring.
 PROOF: $a \Rightarrow b$: Let S be a non-empty set of ideals with no maximal element. Say $I_1 \in S$, then there must exist $I_2 \in S$ with $I_1 \subsetneq I_2$. Then $\exists I_3 \in S$ with $I_1 \subsetneq I_2 \subsetneq I_3$. And so on, contradiction.
 (a). $b \Rightarrow c$: Let I be an ideal in R , let $S = \{J \mid J \text{ is an ideal of } R, J \subseteq I, \text{ and } J \text{ is finitely-generated}\}$. We see $S \neq \emptyset$ as $\{0\} \in S$. By (b), S has a maximal element, J_0 . Let $a \in I$. The ideal $J_0 + \langle a \rangle$ is finitely generated and $\subseteq I$. This forces $J_0 + \langle a \rangle = J_0$ which says $a \in J_0$. This implies $I \subseteq J_0$, so $I = J_0$ and I is finitely generated. break.
 $c \Rightarrow a$: Let $J_1 \subseteq J_2 \subseteq J_3 \subseteq \dots$ be an ascending chain of ideals. We see $J = \bigcup_{i=1}^{\infty} J_i$ is an ideal that by (c) is finitely generated, say by a_1, a_2, \dots, a_n . We know $a_i \in J_{k_i}$. We see that all a_i are in $J_{\max\{k_1, k_2, \dots, k_n\}}$ as is therefore J itself. Says $J_r = J_{r+1} = J_{r+2} = \dots$, where $r = \max\{k_1, k_2, \dots, k_n\}$.

$f_1(x_1, x_2) = 0$
 $f_2(x_1, x_2) = 0$

$A = \langle x \rangle$ in $R[x]$

Let A be an ideal in $R[x]$, for ring R . Let A_n = set of leading coefficients of polynomials in A of degree $\leq n$.

$A = \langle 3x^3 - 1 \rangle$ in $\mathbb{Z}[x]$

We Claim: (a) A_n is an ideal of R . (clear proof), (b) $A_n \subseteq A_{n+1}$ (clear)

$A_0 = \{0\}, A_1 = \{0\}$
 $A_2 = \{0\}, A_3 = \{3\}$

LEMMA: Let A and B be ideals of $R[x]$. If $A \subseteq B$, then $A_n \subseteq B_n \forall n$.

Moreover if $A_n = B_n \forall n$, then $A = B$.

PROOF: That $A_n \subseteq B_n$ is clear. We use induction for the second part.

Note: We call A_n the n^{th} ASSOCIATED IDEAL of A .

Let $f(x) = b_0 + b_1x + \dots + b_nx^n \in B$. If $n=0$, then $f(x) = b_0 \in B_0 = A_0 \subseteq A$.

Now assume that $\leq \deg n-1$ polynomials in B are in A . Look at $f(x)$ of degree n in B . As $f(x) \in B$, we know $b_n \in B_n = A_n$. So \exists polynomial $g(x) \in A$ with $g(x) = a_0 + a_1x + \dots + b_nx^n$. But as $A \subseteq B$, this implies $g(x) \in B$. So $f(x) - g(x) \in B$ and $\deg f(x) - g(x) \leq n-1$. By induction hypothesis, $f(x) - g(x) \in A$. As $g(x) \in A$, we deduce $f(x) \in A$ as desired.

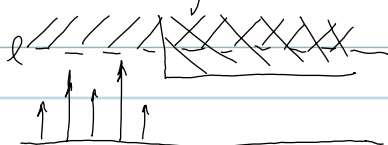
HILBERT'S BASIS THEOREM: If R is a Noetherian ring, so is $R[x]$.

→ PROOF: Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ be an ascending chain of ideals in $R[x]$.

Let A_{ij} be the j^{th} associated ideal of A_i . We have the

looking at the diagonal, we see
 $A_{10} \subseteq A_{21} \subseteq A_{32} \subseteq \dots$ As R is Noetherian, this chain must eventually become constant.

So the diagram looks like the following:



As there are only finitely many columns to the left of the shaded zone, there is some level l past which all vertical chains become constant. By the LEMMA, this says

$A_l = A_{l+1} = A_{l+2} = \dots$, so indeed, $R[x]$ is Noetherian, as it satisfies A.C.C.

COROLLARY: If R is Noetherian, so is $R[x_1, x_2, \dots, x_n]$ for any $n \geq 1$.

PROOF: Use the fact that $R[x_1, x_2, \dots, x_n] = R[x_1, x_2, \dots, x_{n-1}][x_n]$

so $R[x, y] = R[x][y]$.

k a field

J ideal in k

$0 \neq a \in J$

$a^{-1} \cdot a = 1 \in J \Rightarrow J = k$

UPCOMING: Varieties; Algebraic sets

HOMEWORK: Waiting for next week.